Electromagnetism: DC Circuits

FIZIKA SPhO Training

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1 Notes

Circuits are everywhere in daily life. Let's study what powers the device you are reading this on!

1.1 Current, Voltage and Resistance

The basic quantities in circuit analysis at I (current), V (voltage) and R (resistance).

At this point, you should know the basics. **Current** is measured by an **ammeter connected in series,** while **voltage** is measured by a **voltmeter connected in parallel.** You add up resistance normally in series, and take the reciprocal of the sum of reciprocals in parallel. These will be considered trivial and omitted.

1.1.1 Resistivity and Conductivity

Sometimes, it may also be useful to define ρ (resistivity) and σ (conductivity), which follow:

$$R := \frac{\rho L}{A} \tag{1}$$

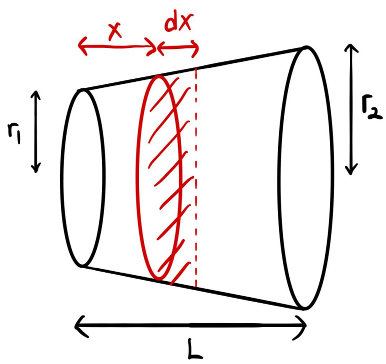
$$\sigma := \frac{1}{\rho} \tag{2}$$

where L is the length of the material and A is the cross-sectional area of the material.

ρ and σ are convenient, because they are **material properties** (i.e. only dependent on the material).

Example 1.1. Calculate the resistance of a circular frustum between the two circular faces. It has radii r_1 and $r_2 > r_1$, length L and resistivity ρ.

We can draw a diagram below to visualise the frustum:



The radius as a function of x is clearly linear. By fitting $r(0) = r_1$ and $r(L) = r_2$, we have

$$r(x) = r_1 + \frac{r_2 - r_1}{L}x$$

Thus, each infinitesimal resistance can be found with Equation (1):

$$dR = \frac{\rho\, dx}{A(x)} = \frac{\rho\, dx}{\pi (r(x))^2} = \frac{\rho\, dx}{\pi \left(r_1 + \frac{r_2 - r_1}{L}x\right)^2}$$

All these infinitesimal resistances add in series. Hence, the total resistance is just the integral:

$$R = \int dR = \int_0^L \frac{\rho\, dx}{\pi \left(r_1 + \frac{r_2 - r_1}{L}x\right)^2} = \frac{\rho L}{\pi r_1 r_2}$$

which is quite a neat result indeed!

1.1.2 Ohm's Law

Ohm's Law directly links the three basic quantities:

$$V := IR \tag{3}$$

In some books, you may instead see Ohm's Law written as

$$\mathbf{J} := \sigma \mathbf{E} \tag{4}$$

Here, $\mathbf{E}$ is just the electric field, while $\mathbf{J}$ is defined as the **current density,** which can just be thought of as current per area.

Clearly, Equation (3) is more useful in circuit analysis, while Equation (4) is more useful when you are finding fields and potentials.

More rigorously, I can be defined as the rate of flow of charge:

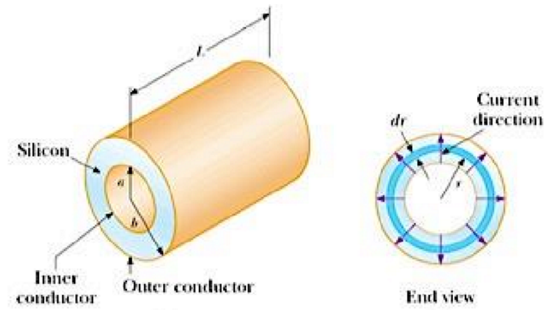
$$I := \frac{dq}{dt} \tag{5}$$

or *even* more rigorously, as the flux of $\mathbf{J}$:

$$I := \oiint_A \mathbf{J} \cdot d\mathbf{A} \tag{6}$$

Remark. Zero current **does not** imply zero voltage, and zero voltage **does not** imply zero current! (Think of examples to verify this.)

Example 1.2 (Ricardo)**.** A coaxial cable consists of two concentric cylindrical conductors of radii a and b. The region in between the conductors is completely filled with silicon of resistivity ρ, and current leaks through in the radial direction as shown in the figure below. Using Ohm's Law, find the effective resistance between the two conductors.



In theory, this example can be done in the same spirit as Example 1.1, by applying Equation (1) and integrating over infinitesimal resistances. However, let's try to apply Ohm's Law.

Suppose a total current I flows radially. The current density σ is hence

$$\sigma = \frac{I}{A} = \frac{I}{2\pi r L}$$

By Ohm's Law,

$$J = \sigma E \quad \Rightarrow \quad E = \frac{J}{\sigma} = \frac{I}{2\pi \sigma r L}$$

Hence, the magnitude of the potential difference is

$$V = \int_a^b E\,dr = \int_a^b \frac{I}{2\pi\sigma r L}\,dr = I\left(\frac{\rho \ln\left(\frac{b}{a}\right)}{2\pi L}\right)$$

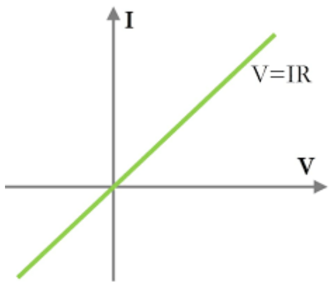
Comparing to the form of Equation (3), this implies that

$$R = \frac{\rho \ln\left(\frac{b}{a}\right)}{2\pi L}$$

1.1.3 I-V Characteristics

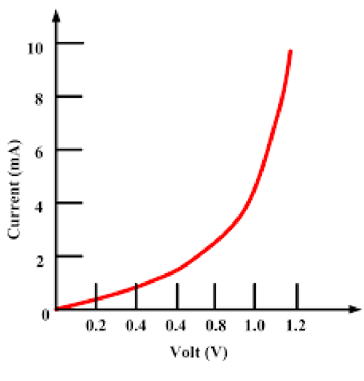
When we analyse circuit elements, we are usually interested in varying I and V. An **I-V characteristic** is a graph of I against V.

For an **Ohmic, linear resistor,** the I-V characteristic is linear, since R is constant in Equation (3):



The resistance is simply just the *inverse* of the gradient of the graph.

For other **non-linear components,** their I-V characteristics may be **curves.** One such example is shown:



In this case, the resistance is not constant and can be calculated at every point by evaluating $\frac{V}{I}$ **at that point.** For the graph above, resistance is decreasing as the gradient is increasing.

Remark. You may be tempted to write Ohm's Law in terms of infinitesimals:

$$R = \frac{dV}{dI}$$

However, this is **wrong** by definition! Ohm's Law gives the relationship between the current and voltage at an instant, and has nothing to do with $\frac{dV}{dI}$ (which is meaningless).

1.1.4 Electromotive Force (Emf)

Interestingly, **emf is NOT a force,** even though its name has "force" in it!

The emf, ε, is the work done by a source to drive a unit charge around a circuit:

$$\varepsilon := \frac{dW}{dq} \tag{7}$$

Remark. Emf is often confused with voltage. While both have units of volts (V), emf is a characteristic of the **source,** while voltage is measured across any two points in a circuit!

We will revisit emf when we discuss electromagnetic induction. For now, it suffices to think of it as the voltage across a source.

1.1.5 Power Dissipated by Circuit Elements

For any circuit element, the electric power dissipated is

$$P := VI \tag{8}$$

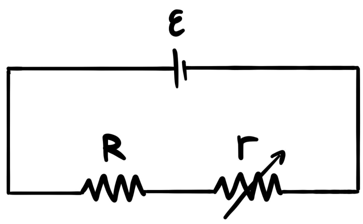
Only for Ohmic resistors, we may write

$$P = \frac{V^2}{R},\ P = I^2 R \tag{9}$$

Another thing to note is that the **total power** dissipated in a circuit (this includes power dissipated by sources, which is usually negative) is 0.

1.1.6 Maximum Power Transfer Theorem

Consider the following circuit with a battery of emf ε and two resistors R and r connected in series. Suppose R is fixed, and r can be freely adjusted.



Suppose we want to find r such that the power transferred to it is maximum.

$$P_r = I^2 r = \left(\frac{\varepsilon}{R+r}\right)^2 r = \frac{\varepsilon^2 r}{(R+r)^2} \tag{10}$$

At maximum power, $\frac{dP_r}{dr} = 0$. Thus,

$$\frac{dP_r}{dr} = \varepsilon^2 \frac{(R+r)^2 - 2r(R+r)}{(R+r)^4} = \varepsilon^2 \frac{R-r}{(R+r)^3} = 0 \tag{11}$$

This implies that maximum power transfer occurs when

$$r_{max\,power} = R \tag{12}$$

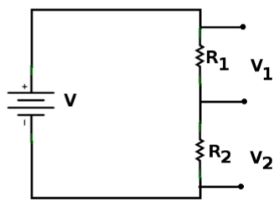
Equation (10) is simply the **maximum power transfer theorem.** The power transferred to a load resistance is maximum when it is equal to the source resistance in series with it.

1.2 Circuit Analysis

In circuit analysis, our objective is to solve for currents in branches of circuits, or the potential differences between circuit elements.

1.2.1 Voltage and Current Divider Rules

Consider the following circuit:



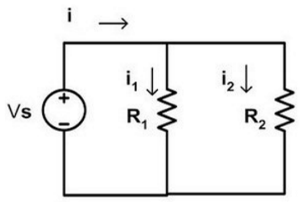
The **voltage divider rule** states that

$$V_1 = \frac{R_1}{R_1 + R_2}V, \; V_2 = \frac{R_2}{R_1 + R_2}V \tag{13}$$

or, in general,

$$V_i = \frac{R_i}{\sum R}V \tag{14}$$

Consider another circuit:



The **current divider rule** states that

$$I_1 = \frac{R_2}{R_1 + R_2}I, \; I_2 = \frac{R_1}{R_1 + R_2}I \tag{15}$$

or, in general,

$$I_i = \frac{R_i}{\sum \frac{1}{R}}I \tag{16}$$

Remark. In practice, you don't need these rules, and you can just work everything out with Ohm's Law. But hopefully these rules help to simplify and speed up your calculations.

1.2.2 Kirchhoff's Laws

Kirchhoff's Laws are the fundamentals to circuit analysis. There are two laws - **Kirchhoff's Current Law (KCL)** and **Kirchhoff's Voltage Law (KVL).**

KCL: The sum of currents entering/leaving a node (a point in the circuit) is 0. Mathematically,

$$\sum I = 0 \tag{17}$$

The assignment of signs (for entering/leaving) is arbitrary, as long as you are consistent. In this handout, we shall assign outgoing currents as positive and incoming currents as negative.

KVL: The sum of potential **drops** across a loop is 0. Mathematically,

$$\sum \Delta V = 0 \tag{18}$$

The assignment of signs is positive for potential drops (positive to negative) and negative for potential gains (negative to positive).

1.2.3 Nodal Analysis

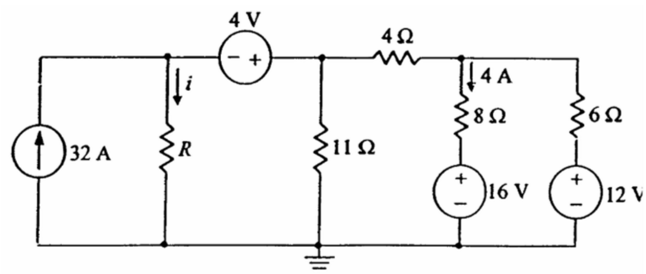
In **nodal analysis,** we turn our attention to the **nodes** in a circuit. In particular, we are interested in the **nodal voltages.**

The procedure for nodal analysis is as follows:

1. Since potentials are physically meaningless (only potential differences matter), we arbitrarily assign the ground (0 voltage) to some nodes.
2. After which, we assign the other nodal voltages in terms of unknowns.
3. Then, we can find the currents based on the nodal voltages and resistances.
4. Finally, using KCL, we can set up simultaneous equations to solve for the unknowns.

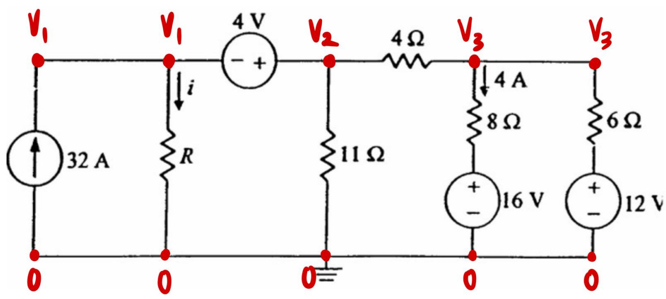
The example below illustrates how to use nodal analysis.

Example 1.3 (Ricardo)**.** Determine i and R in the circuit below.



We start by grounding some nodes. Since the bottom 5 nodes are all connected by wires, grounding them will make the most points have 0 voltage, so this is the best choice.

Now, we label the other nodal voltages in terms of unknowns:



It is easiest to apply KCL to the right nodes at V_3:

$$4 + \frac{V_3 - V_2}{4} + \frac{V_3 - 12}{6} = 0$$

Since the points of V_1 and V_2 are joined by a 4V voltage, we have:

$$V_2 = V_1 + 4$$

The last KCL equation appears a bit complicated. However, this can be fixed if we treat the nodes of V_1 and V_2 directly adjacent to each other as a **supernode!** We can then consider KCL on this entire supernode, giving:

$$-32 + \frac{V_1}{R} + \frac{V_2}{11} + \frac{V_2 - V_3}{4} = 0$$

The above three equations can be solved simultaneously to obtain

$$V_1 = 84\,\text{V},\ R = 6\,\Omega \quad \Rightarrow \quad i = \frac{V_1}{R} = 14\,\text{A}$$

1.2.4 Mesh Analysis

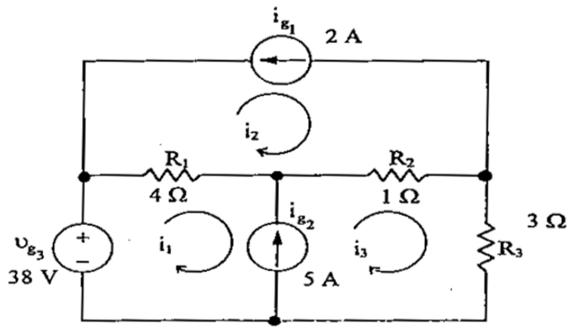
In **mesh analysis,** we turn our attention to the **meshes** in a circuit, which are the "loops". In particular, we are interested in the **mesh currents.**

The procedure for mesh analysis is as follows:

1. Assign an unknown to the current flowing through each mesh. It is recommended to assign **clockwise** mesh currents.
2. Find the current through each circuit component, taking into account the signs of the mesh currents through it.
3. Finally, using KVL, we can set up simultaneous equations to solve for the unknowns.

The example below illustrates how to use mesh analysis.

Example 1.4 (Ricardo)**.** Determine the mesh currents i_1, i_2 and i_3 in the circuit below.



Referring to the mesh currents above, we can simply see that

$$i_2 = -2\,\text{A}$$

The **negative sign** is very important as we defined our mesh current's direction opposite of the 2 A current source!

The other KVL equation appears a bit complicated. However, this can be fixed if we treat the two bottom two meshes directly adjacent to each other as a **supermesh!** We can then consider KVL on this entire supermesh, giving:

$$-38 + 4i_1 + 4i_3 - 5i_2 = 0$$

Lastly, don't forget that i_1 and i_3 are constrained by the 5 A current source:

$$i_3 - i_1 = 5$$

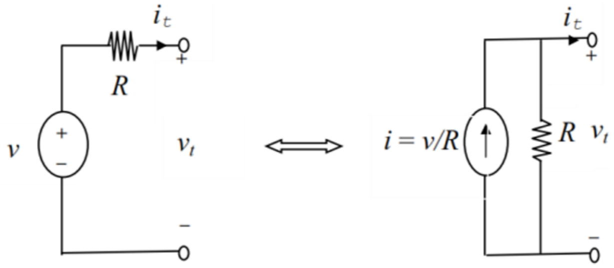
The above three equations can be solved simultaneously to obtain

$$i_1 = 1\,\text{A},\ i_2 = -2\,\text{A},\ i_3 = 6\,\text{A}$$

1.2.5 Source Transformations

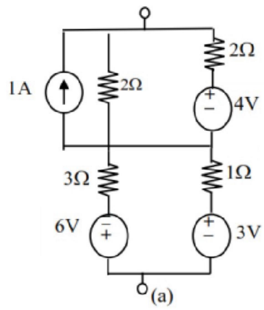
In the above examples, you've seen many constant voltage sources and constant current sources. Sometimes, it may be more useful to transform one into another. We call this the **source transformations.**

The fundamental idea is that the following two circuits are equivalent:

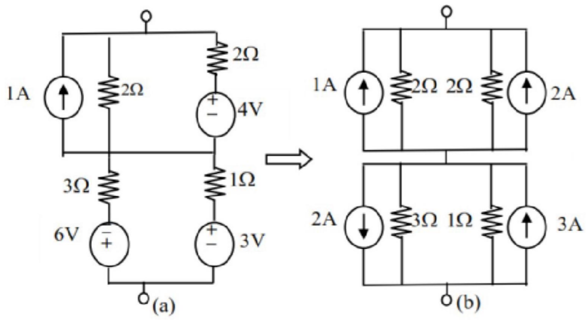


due to Ohm's Law. Source transformations are useful as they allow us to simplify complicated circuits. The example below illustrates.

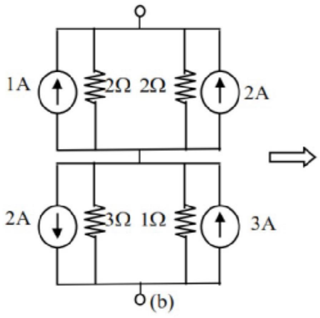
Example 1.5 (Ricardo)**.** Simplify the following circuit into one with only one resistor and one current/voltage source.



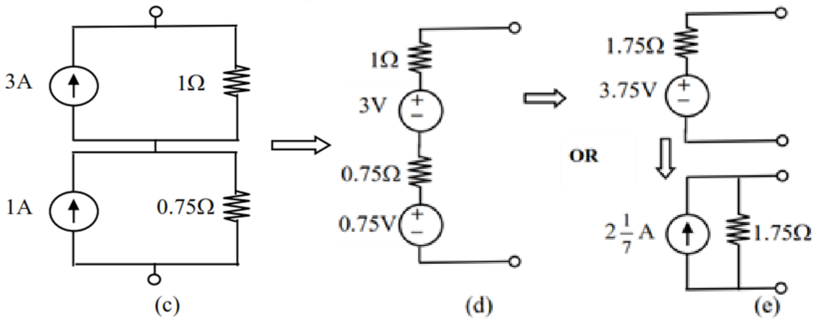
It is more convenient to convert all the sources to the same type. Let's convert all the voltage sources into current sources:



We can then deal with each part of the circuit separately:



At this point, it is easier to convert back to voltage sources to combine them together:



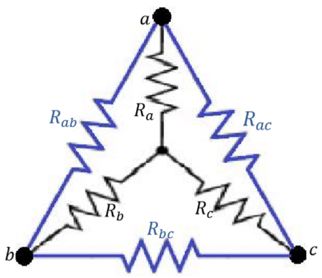
hence giving our final circuit.

1.3 Ideas

Many tricky electromagnetism problems involve the use of the following ideas.

1.3.1 Δ-Y and Y-Δ Transformations

Consider three resistors, either in a Δ-shaped arrangement or a Y-shaped arrangement below:



The **Δ-Y and Y-Δ transformations** relate the resistances in the Δ configuration (in blue) and the Y configuration (in black):

$$R_a = \frac{R_{ab}R_{ac}}{R_{ab}+R_{bc}+R_{ac}}, \quad R_b = \frac{R_{ab}R_{bc}}{R_{ab}+R_{bc}+R_{ac}}, \quad R_c = \frac{R_{ac}R_{bc}}{R_{ab}+R_{bc}+R_{ac}} \tag{19}$$

$$R_{ab} = \frac{R_aR_b+R_bR_c+R_aR_c}{R_c}, \quad R_{bc} = \frac{R_aR_b+R_bR_c+R_aR_c}{R_a}, \quad R_{ac} = \frac{R_aR_b+R_bR_c+R_aR_c}{R_b} \tag{20}$$

In the special case where all three resistances are equal, we have

$$R_\Delta = 3R_Y \tag{21}$$

The proof of these transformations don't require anything beyond knowing resistors in series and parallel. (Prove them yourself as an exercise!)

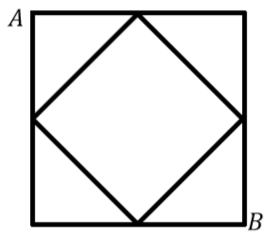
Remark. Often, the Δ-Y transformation is more useful, since resistor networks involving Δs are hard to deal with directly.

1.3.2 Equipotential Points and Symmetry

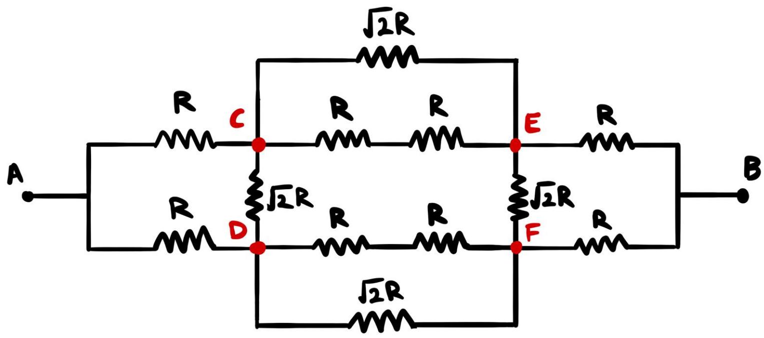
When simplifying complex circuits, spotting **symmetry** is very important. It allows you to identify **equipotential points** (points at the same potential), which can simplify whole branches of circuits!

The example below illustrates how to spot and utilise equipotential points.

Example 1.6 (SJPO Special Round 2009)**.** The shape shown in the figure is made of wire of constant cross section. The side of the bigger square is a m, and a 1 m length of wire has a resistance ρ. Determine the resistance between points A and B.

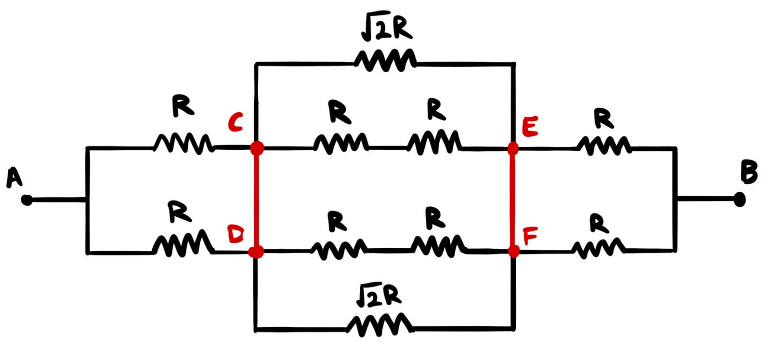


We can first simplify this by drawing out the circuit diagram. Let $R = \frac{\rho a}{2}$:



Notice that there is a high level of symmetry here. The points C and D are symmetric, and so are the points E and F. As such, there shouldn't be a preference for current to flow in either direction, so **no current flows** between these two pairs of points! (We call C, D and E, F pairs of equipotential points.)

As such, we can completely ignore the $\sqrt{2}R$ resistances between these pairs of points, and effectively join them with a wire (so that they are at the same potential):



This circuit is easy to evaluate as it is just series/parallel combinations. You should obtain:

$$R_{AB} = \frac{R}{2} + \frac{1}{2\left(\frac{1}{\sqrt{2}R}\right) + 2\left(\frac{1}{2R}\right)} + \frac{R}{2} = \sqrt{2}R = \frac{\sqrt{2}\rho a}{2}$$

1.3.3 Current Injection and Superposition

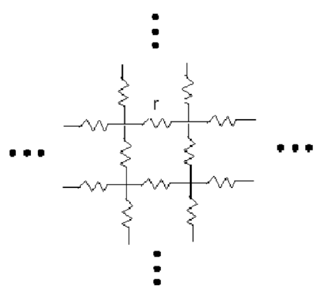
When dealing with even more complicated networks, it may be useful to consider two separate scenarios:

1. Put current I into one node.
2. Extract current I from another node.

We can analyse these scenarios separately to find the potential differences we are interested in. Afterwards, using superposition, we can "add" these scenarios together. Finally, using $V = IR_{eff}$ and equating the V found from superposition, we can determine R_{eff}.

This may sound a little confusing, but the example below should illustrate what it means.

Example 1.7 (SPhO 2003)**.** Consider the infinite square array of resistors each of resistance r as shown in the figure. Find the equivalent resistance between two neighbouring points separated by a resistor r.



Let's focus our attention on the points left and right to the resistor labelled r.

Suppose a current I flows in via the left point. By symmetry, a current $\frac{I}{4}$ flows through each of the resistors connected to that point. This contributes a potential difference of

$$V_{left} = \frac{Ir}{4}$$

between the two points.

Now, suppose we extract a current I via the right point. By symmetry, a current $\frac{I}{4}$ flows through each of the resistors connected to that point. This contributes a potential difference of

$$V_{right} = \frac{Ir}{4}$$

between the same two points.

We can then superpose the two cases together. By doing so, we simply add up the potential differences:

$$V = V_{left} + V_{right} = \frac{Ir}{2}$$

At the same time, $V = IR_{eff}$. Hence,

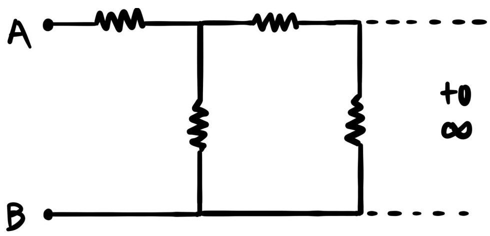
$$IR_{eff} = \frac{Ir}{2} \quad \Rightarrow \quad R_{eff} = \frac{r}{2}$$

Remark. The last step is true, because by superposing the two cases together, we are essentially viewing the circuit as a constant current source of I connected between the two points.

1.3.4 Infinite Networks

Another common type of Olympiad question is the **infinite network.** To solve such problems, you want to use the concept of **self-similarity** (i.e. the circuit contains itself).

Example 1.8. Determine the effective resistance between the two terminals of the infinite resistor ladder as shown below. Each resistor has a resistance R.



Notice that since the ladder repeats to infinity, you can replace a part of the ladder with itself. Suppose the effective resistance is R_{eff}. Then, R_{eff} is the same as having R in series with a resistance of R in parallel with R_{eff}.

In particular, the following equation is satisfied:

$$R_{eff} = R + \frac{RR_{eff}}{R + R_{eff}} \quad \Rightarrow \quad R_{eff}^2 - RR_{eff} - R^2 = 0$$

We can solve this quadratic equation, and take the (physically meaningful) positive root to get:

$$R_{eff} = \left(\frac{1+\sqrt{5}}{2}\right) R$$

1.3.5 Negative Resistance

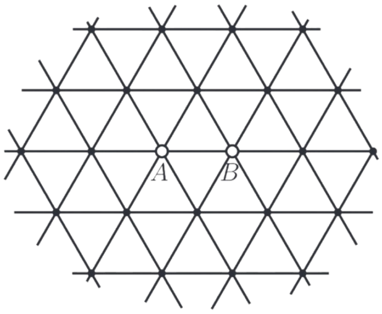
The last concept is negative resistance. This is especially useful in a situation that has resistors cut off, but would have otherwise been highly symmetric.

Essentially, you want to treat "missing resistances" as negative. There are two possibilities:

1. **Connect R and $-R$ in series:** This will give 0 resistance (a short circuit).
2. **Connect R and $-R$ in parallel:** This will give ∞ resistance (an open circuit).

The example below illustrates.

Example 1.9 (Kalda)**.** (i) Determine the effective resistance between two neighbouring nodes in an infinite triangular lattice, as shown in the figure below. The resistance of the wire connecting any two neighbouring nodes is R. (ii) Now, suppose the wire connecting nodes A and B is cut. Determine the new effective resistance.



(i) We shall solve the first part in the spirit of Example 1.7.

Suppose a current I flows in via A. By symmetry, a current $\frac{I}{6}$ flows through each of the wires connected to A. This contributes a potential difference of

$$V_A = \frac{IR}{6}$$

between points A and B.

Now, suppose we extract a current I via B. By symmetry, a current $\frac{I}{6}$ flows through each of the wires connected to B. This contributes a potential difference of

$$V_B = \frac{IR}{6}$$

between points A an B.

Superposing the two cases together,

$$V = V_A + V_B = \frac{IR}{3}$$

At the same time, $V = IR_{eff}$. Hence,

$$IR_{eff} = \frac{IR}{3} \quad \Rightarrow \quad R_{eff} = \frac{R}{3}$$

(ii) Notice that all the symmetry is broken by the cut wire. This inspires us to think about connecting a negative resistance.

Imagine putting two resistors of R and $-R$ in parallel between A and B, in place of the cut wire. This has the same effect as cutting the wire, since the effective resistance is ∞.

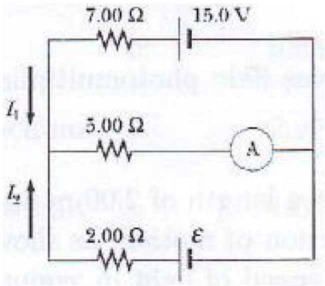
The circuit now becomes $\frac{R}{3}$ and $-R$ in parallel. Thus,

$$R_{eff} = \frac{\frac{1}{3}(-1)}{\frac{1}{3} - 1} R = \frac{R}{2}$$

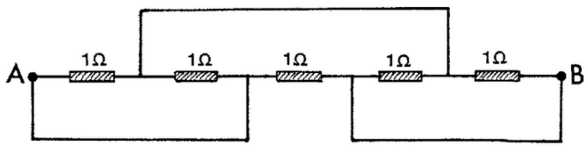
2 Problems

Problems are arranged in roughly increasing difficulty.

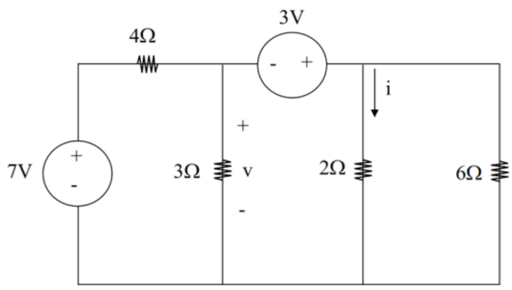
Problem 2.1 (SPhO 2009)**.** Two ideal cells and three resistors with resistances 2.00, 5.00 and 7.00 Ω are connected as shown in the figure. If the ammeter reads 2 A, determine the currents I_1, I_2 and determine ε.



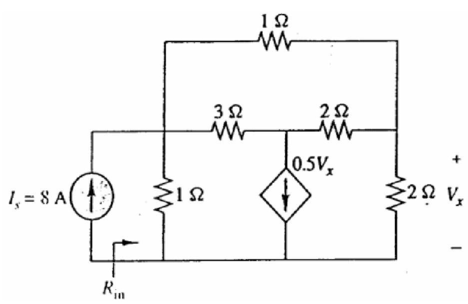
Problem 2.2 (IPhO 1996)**.** Five 1 Ω resistances are connected as shown in the figure below. The resistance in the wires is negligible. Determine the effective resistance between A and B.



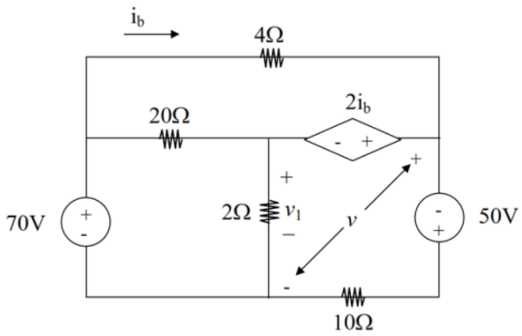
Problem 2.3 (Ricardo)**.** Find v and i in the circuit shown below.



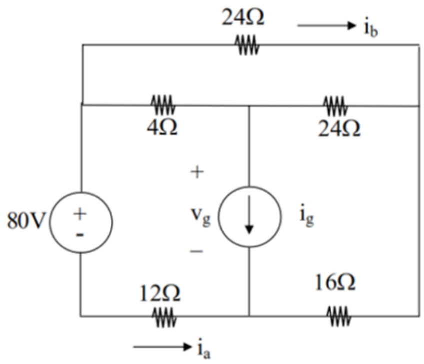
Problem 2.4 (Ricardo)**.** For the circuit shown, determine V_X and the equivalent input resistance R_{in} seen at the current source.



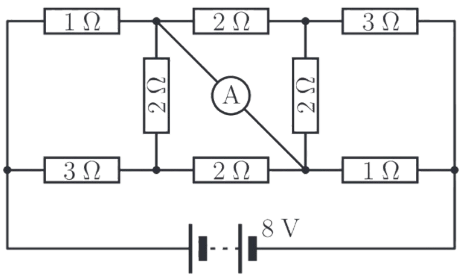
Problem 2.5 (Ricardo)**.** Determine the voltage v in the circuit shown below.



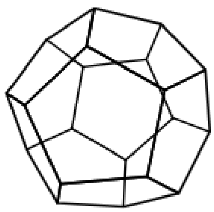
Problem 2.6 (Ricardo)**.** The currents i_a and i_b in the circuit shown below are 4 A and 2 A respectively. Show that the power delivered by the current source is equal to the power absorbed by all the other elements. (This proves that energy is conserved.)



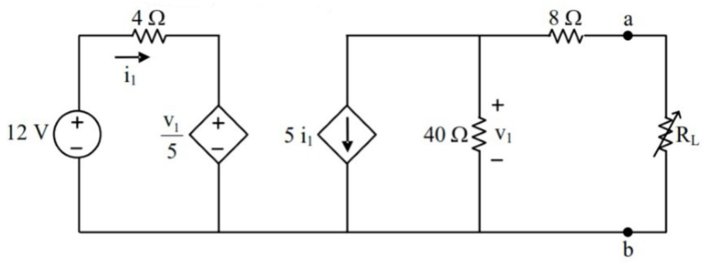
Problem 2.7 (Kalda)**.** Determine the reading of the ideal ammeter in the circuit below.



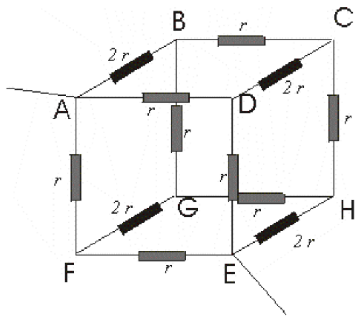
Problem 2.8 (Kalda)**.** (i) Determine the resistance between two neighbouring vertices of a dodecahedron (see figure). The edges are made of wire and the resistance of each edge is R. (ii) Now, the wire joining neighbouring vertices A and B on the dodecahedron is cut off. Determine the resistance between A and B.



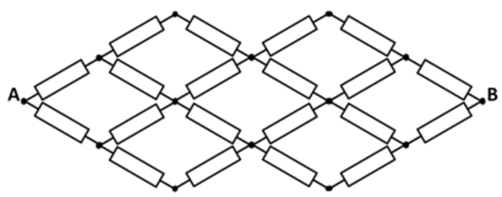
Problem 2.9 (Ricardo)**.** Find the value of R_L that will draw the maximum power from the rest of the circuit. Calculate the corresponding maximum power consumed by R_L.



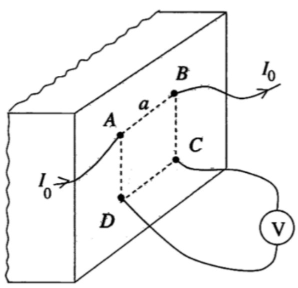
Problem 2.10 (SPhO 2002)**.** *ABCDEFGH* is a cuboid with resistances $2r$ along the edges AB, CD, FG and EH and resistances r along the rest of its edges. Find the equivalent resistance of the circuit between the points A and E as shown in the figure.



Problem 2.11 (SJPO Special Round 2014)**.** Twenty identical resistors of resistance r are connected as shown in the figure below. What is the equivalent resistance between A and B?



Problem 2.12 (200 Puzzling Physics Problems)**.** A plane divides space into two halves. One half is filled with a homogeneous conducting medium, and physicists work in the other. They mark the outline of a square of side a on the plane and let a current I_0 in and out at two of its neighbouring corners. Meanwhile, they measure the potential difference V between the two other corners. Find the resistivity ρ of the medium.

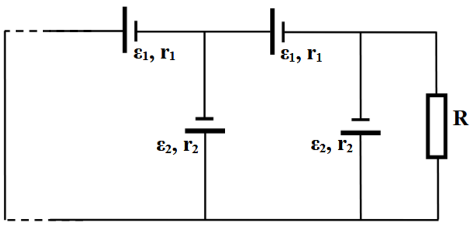


3 Advanced Problems

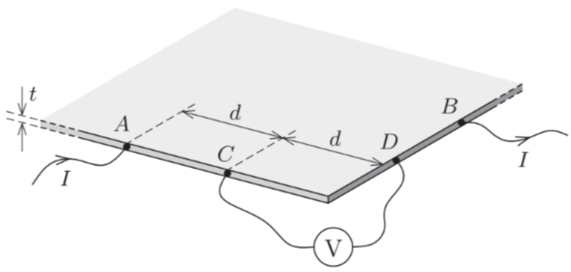
These problems are way too difficult to be tested in a modern-day SPhO. If you have completed all the previous problems and are down for a challenge, try these!

Problem 3.1 (SPhO 2015)**.** There are 2015 points, each connected to all other points by a resistor of resistance R. Find R_{total} between any 2 points.

Problem 3.2 (IZhO 2011)**.** The resistance $R = 2.0\,\Omega$ is connected to an infinite number of voltage sources obtained by repetition as shown in the figure below. Find the current flowing through the resistance R. The emfs and internal resistances of the sources are known to be $\varepsilon_1 = 2.0\,\text{V}$, $r_1 = 1.0\,\Omega$ and $\varepsilon_2 = 1.0\,\text{V}$, $r_2 = 2.0\,\Omega$.



Problem 3.3 (200 Puzzling Physics Problems)**.** We aim to measure the resistivity of the material of a large, thin, homogeneous square metal plate, of which only one corner is accessible. To do this, we choose the points A, B, C and D on the side edges of the plate that form the corner.



Points A and B are both $2d$ from the corner, whereas C and D are each a distance d from it. The length of the plate's sides is much greater than d, which is in turn much greater than the thickness t of the plate. If a current I enters the plate at point A, and leaves it at B, then the reading on a voltmeter connected between C and D is V. Find the resistivity ρ of the plate material. *Hint: This problem is different from Problem 2.12! It is not as easy.*